THE KING'S SCHOOL

2006
Higher School Certificate
Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1-7
- All questions are of equal value
- Start a new booklet for each Question
- Put your Student Number and the Question Number on the front of each booklet

Total marks – 84
Attempt Questions 1-7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate $y = \log_e(\cos x)$ expressing your answer in simplest form. **2**

(b) Evaluate:

(i) $\displaystyle\int_2^3 \frac{2x}{\sqrt{x^2 - 4}}\,dx$ using the substitution $u = x^2 - 4$. **3**

(ii) $\displaystyle\int_{\pi}^{\frac{4\pi}{3}} \sin x \cos x \, dx$ **3**

(c) Solve the following inequaltiy for x, graphing the solution on a number line

$$\frac{1}{x+2} < 3$$ **2**

(d) Determine the acute angle between the straight lines whose equations are

$$x - y + 1 = 0 \text{ and } 2y = x + 1$$ **2**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $y = 2\cos^{-1}\dfrac{x}{3}$.

(i) Sketch the graph of this function clearly showing the domain and range. **2**

(ii) Find the angle, θ, that the tangent to the curve $y = 2\cos^{-1}\dfrac{x}{3}$ at $x = 0$ makes with the positive direction of the x-axis. **3**

(b) Find the volume of the solid of revolution formed when the curve $y = \sin x$ is rotated around the x-axis between the lines $x = 0$ and $x = \dfrac{\pi}{4}$. **3**

(c) Find all values of θ for which $2\sin\theta - \sqrt{2} = 0$. **2**

(d) Find the point P which divides the interval joint A(-3, 5) and B(7,10) externally in the ratio 2 : 7. **2**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A spherical balloon is expanding so that its volume V mm^3 increases at a constant rate of 72 mm^3 per second.

What is the rate of increase of its surface area A mm^2, when the radius is 12 mm? **4**

(b) Factorise $x^3 - 3x^2 - 10x + 24$, given that $x = 2$ is a zero, and hence solve $x^3 + 24 > 3x^2 + 10x$. **4**

(c) Solve $3\sin x + 2\cos x = 2$, for $0 \leq x \leq 360^\circ$, to the nearest minute. **4**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

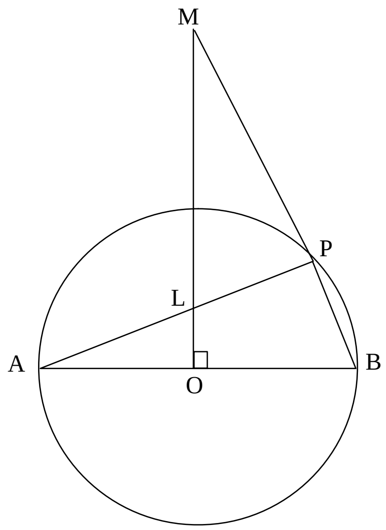
(a) Find the term independent of x in the expansion of $\left(x + \frac{1}{x^2}\right)^9$. **3**

(b) Consider the expansion of $(1 + 2x)^n$.

(i) Write down an expression for the coefficient of the term in x^4. **2**

(ii) The ratio of the coefficient of the term in x^4 to that of the term in x^6 is 5 : 8. Find n. **3**

(b)



O is the centre of the circle, MPB is a straight line and OLM is perpendicular to AOB as shown. Prove that:

(i) A, O, P, M are concyclic, and **2**

(ii) $\angle$ OPA $=$ $\angle$ OMB. **2**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) TA and TB are two tangents drawn to a circle from an external point T. A and B are the point of contact of the tangent with the circle.

(i) Draw a neat diagram clearly showing this information.

(ii) Prove that $TA = TB$. **2**

(b) Prove by Mathematical Induction that $7^n - 1$ is divisible by 6 for all positive integers of n. **4**

(c) The elevation of hill at a place A due east of it is 39°, at a place B due south of A, the elevation is 27°.

If the distance from A to B is 500m, find the height of the hill, to the nearest metre. **4**

(d) Show that $\frac{d}{dx}(\log_e 2x) = \frac{d}{dx}(\log_e x)$.
Does this mean that $\log_e 2x = \log_e x$?

Give reasons for your answers. **2**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Consider the parabola $y = x^2$.

Find the equation of the tangent to the parabola at the point $P(t, t^2)$. **2**

(ii) Show that the line passing through the focus of the parabola and perpendicular to the tangent at P had equation $x = \frac{t}{2}(1 - 4y)$. **2**

(iii) Find the locus of $Q\ (X, Y)$, the point of intersection of the tangent and the line through the focus perpendicular to the tangent. **2**

(b) A particle *A* is projected horizontally at 50 *m/s* from the top of a tower 100m high. At the same instant, another particle *B* is projected from the bottom of the tower, in the same vertical plane at 100 *m/s* with elevation $60°$.

Prove that the particles will collide and find where they do so. (Use $g = 10ms^{-2}$.) **6**

End of Question 6

Question 7 appears on next page

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve for $0^\circ \le \theta \le 360^\circ$, $\sin x = 3\cos(x + 65^\circ)$. **3**

(b) Prove that the area of a ΔABC is $\dfrac{a^2 \sin B \sin C}{2 \sin A}$. **2**

(c) Given that $y = \dfrac{x^2 + \lambda}{x + 2}$ and x is real, find:

(i) the set of value(s) of λ for which y can take all but one real value. **1**

(ii) If when $\lambda = 5$, by sketching $y = \dfrac{x^2 + 5}{x + 2}$, find the range of the function. **3**

(d) Find the values of m for which the line $y = mx$ touches the curve $y = \dfrac{2x^2 + 1}{2(x + 2)}$. **3**

End of Examination

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

Note: $\ln x = \log_e x, \quad x > 0$

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1(a) $y = \log_e(\cos x)$

$y' = \dfrac{-\sin x}{\cos x}$

$= -\tan x$

(b)(i) $\displaystyle\int_2^3 \frac{2x\,dx}{\sqrt{x^2-4}}$ $u = x^2 - 4$

$\dfrac{du}{dx} = 2x$

$x = 2,\ u = 0$

$x = 3,\ u = 5$

$I = \displaystyle\int_0^5 \frac{2x}{\sqrt{u}} \times \frac{du}{2x}$

$= \displaystyle\int_0^5 u^{-\frac{1}{2}}\,.du$

$= \left[2u^{\frac{1}{2}}\right]_0^5$

$= 2\sqrt{5}.$

(ii) $\displaystyle\int_{\pi}^{\frac{4\pi}{3}} \sin x \cos x\,dx$

$= \displaystyle\int_{\pi}^{\frac{4\pi}{3}} \tfrac{1}{2}\sin 2x\,dx$

$= \left[-\tfrac{1}{4}\cos 2x\right]_{\pi}^{\frac{4\pi}{3}}$

$= -\tfrac{1}{4}\cos\frac{8\pi}{3} + \tfrac{1}{4}\cos 2\pi$

$= -\tfrac{1}{4}\cos\frac{2\pi}{3} + \tfrac{1}{4}$

$= \tfrac{1}{8} + \tfrac{1}{4}$

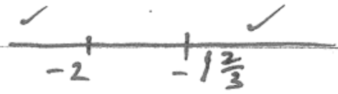
$= \tfrac{3}{8}.$

(c) $\dfrac{1}{x+2} < 3$ $x \neq -2$

C.P. $x = -2$

$1 = 3(x+2)$

$x = -1\frac{2}{3}$



test $x = 0$ $\frac{1}{2} < 3$ true

$\therefore x < -2$ or $x > -1\frac{2}{3}$.

(d) $l_1 : m_1 = 1$

$l_2 : m_2 = \frac{1}{2}$

$\tan\theta = \left|\dfrac{1 - \frac{1}{2}}{1 + 1\times\frac{1}{2}}\right|$

$= \left|\dfrac{\frac{1}{2}}{\frac{3}{2}}\right|$

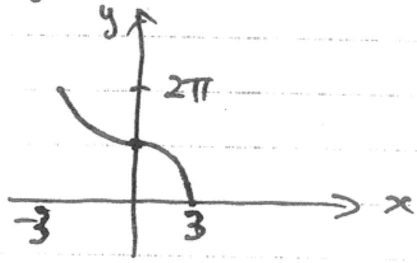
$= \frac{1}{3}$

$\therefore \theta = 18^\circ 26'$

2(a) $y = \cos^{-1} x$ $D: -1 \le x \le 1$

$R: 0 \le y \le \pi$

(i) $y = 2\cos^{-1}\frac{x}{3}$



(ii) $y = 2\cos^{-1}\frac{x}{3}$

$y' = 2 \times \frac{-1}{\sqrt{1-\frac{x^2}{9}}} \times \frac{1}{3}$

$= \frac{-2}{\sqrt{9-x^2}}$

at $x = 0$ $y' = -\frac{2}{3}$

$\therefore \tan\theta = \frac{-2}{3}$

$\theta = 146°19'$

(b) $V = \pi\int_0^{\frac{\pi}{4}} \sin^2 x \cdot dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{4}}(1-\cos 2x)dx$

$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} - (0)\right]$

$= \frac{\pi}{2} \times \left(\frac{\pi}{4} - \frac{1}{2}\right)$

$= \frac{\pi(\pi-2)}{8}$ u^3

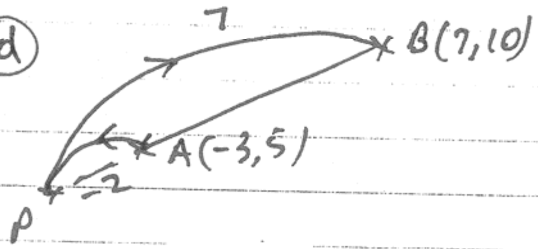
(c) $2\sin\theta - \sqrt{2} = 0$

$\sin\theta = \frac{\sqrt{2}}{2}$

$= \frac{1}{\sqrt{2}}$

$\theta = n\pi + (-1)^n\frac{\pi}{4}$

(d)



$x = \frac{-14-21}{7-2}$ $y = \frac{-20+35}{7-2}$

$= -\frac{35}{5}$ $= \frac{15}{5}$

$= -7$ $= 3$

$P(-7, 3)$

Q 3 (a) $\frac{dV}{dt} = 72$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= \frac{1}{4\pi \times 12^2} \times 72$

$= \frac{1}{8\pi}$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 8\pi \times 12 \times \frac{1}{8\pi}$

$= 12 \text{ mm}^2/\text{sec}.$

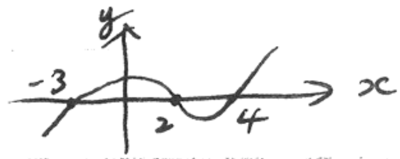
(b)

$$\begin{array}{r|l} & x^2 - x - 12 \\ x-2 & x^3 - 3x^2 - 10x + 24 \\ & x^3 - 2x^2 \\ & \quad -x^2 - 10x \\ & \quad -x^2 + 2x \\ & \qquad -12x + 24 \\ & \qquad -12x + 24 \\ & \qquad \cdot \end{array}$$

$x^3 - 3x^2 - 10x + 24 = (x-2)(x-4)(x+3)$

$x^3 + 24 > 3x^2 + 10x$

$x^3 - 3x^2 - 10x + 24 > 0$



$-3 < x < 2$ or $x > 4$

(c) $3\sin x + 2\cos x = 2$

$3\sin x + 2\cos x$

$\equiv R\sin(x+\alpha)$

$\equiv R[\sin x.\cos\alpha + \cos x.\sin\alpha]$

$R = \sqrt{3^2+4^2} = \sqrt{13}$

$\cos\alpha = \frac{3}{\sqrt{13}}$

$\alpha = 33°41'$

$\therefore \sqrt{13}\sin(x + 33°41') = 2$

$\sin(x + 33°41') = \frac{2}{\sqrt{13}}$

$\therefore x + 33°41' = 33°41', 146°18'$ or $393°41'$

$\therefore x = 0°, 112°38'$ or $360°$

or

$3\left(\frac{2t}{1+t^2}\right) + 2\left(\frac{1-t^2}{1+t^2}\right) = 2$

$6t + 2 - 2t^2 = 2 + 2t^2$

$2t^2 - 3t = 0$

$t(2t-3) = 0$

$t = 0$ or $\frac{3}{2}$

$\tan\frac{x}{2} = 0$ or $\frac{3}{2}$

$\frac{x}{2} = 0°, 180°$ or $56°19'$

$x = 0°, 360°$ or $112°38'$

Q4) ⓐ

General Term $= {}^9C_r\, x^r \cdot \left(\frac{1}{x^2}\right)^{9-r}$

$= {}^9C_r \dfrac{x^r}{x^{18-2r}}$

$= {}^9C_r \cdot x^{3r-18}$

$\therefore 3r-18 = 0$

$r = 6$

Term is ${}^9C_6 = 84$.

ⓑ (i) $(2x+1)^n$

General term $= {}^nC_r (2x)^r \cdot 1^{n-r}$

$= {}^nC_r (2x)^r$

Term in x^4 $r=4$

$\therefore$ coeff $= {}^nC_4 \cdot 2^4$

(ii) Term in x^6 $r=6$

coeff $= {}^nC_4 \cdot 2^6$

$\therefore \dfrac{{}^nC_4 \cdot 2^4}{{}^nC_6 \cdot 2^6} = \dfrac{5}{8}$

$$\frac{\dfrac{n!}{4!(n-4)!}}{\dfrac{n!}{6!(n-6)!} \cdot 2^2} = \frac{5}{8}$$

$$\frac{\cancel{n!}}{4!(n-4)!} \times \frac{6!(n-6)!}{\cancel{n!}} = 4 \times \frac{5}{8}$$

$$\frac{6 \times 5}{(n-4)(n-5)} = \frac{5}{2}$$

$12 = n^2 - 9n + 20$

$n^2 - 9n + 8 = 0$

$(n-1)(n-8) = 0$

$n \neq 1 \therefore n = 8$

ⓒ (i) $\angle AOM = 90°$ (data)

$\angle APB = 90°$ (angle in semicircle)

$\angle APM = 90°$ (st. line)

$\therefore$ A, O, M, P as

$\angle AOM = \angle APM$

(angles in same seg on circle)

(ii)

$\angle OAP = \angle OMP$

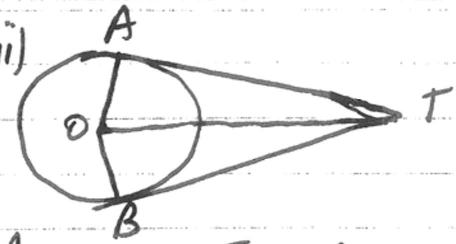
(angles in same seg)

$\angle OAP = \angle OPA$

(base angles of isos $\triangle$)

$\therefore \angle OPA = \angle OMP$

Q5 (a) (ii)



In Δ's OAT & OBT

OA = OB (radii of circle)

OT is common

$\angle OAT = \angle OBT = 90°$

(tang & radius make 90°)

$\therefore$ Δ's congruent (RHS)

$\therefore$ TA = TB

(corresponding sides of congruent triangles)

(b) when $n = 1$ $\quad 7^1 - 1$ is divisible by 6

assume true for $n = k$

i.e. $\dfrac{7^k - 1}{6} = M$ where M is +ve integer

$\therefore 7^k = 6M + 1$

prove true for $n = k+1$

i.e. $\dfrac{7^{k+1} - 1}{6} = M_1$ (a +ve integer)

$$LHS = \frac{7 \times 7^k - 1}{6}$$

$$= \frac{7 \times (6M+1) - 1}{6}$$

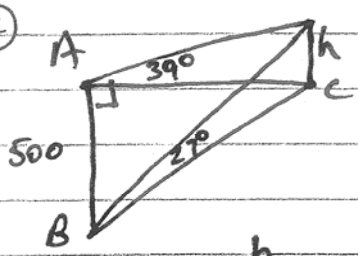
$$= \frac{42M + 7 - 1}{6}$$

$$= 7M + 1 \text{ (a positive integer)}$$

$\therefore$ if true for $n = k$, it is true for $n = k+1$

since it is true for $n = 1$, it is true for $n = 2$ and so on for any positive integer

Q5 (c)



$$\tan 39° = \frac{h}{AC}$$
$$AC = \frac{h}{\tan 39°}$$

$$\tan 27° = \frac{h}{BC}$$
$$BC = \frac{h}{\tan 27°}$$

$$BC^2 = 500^2 + AC^2$$
$$\frac{h^2}{\tan^2 27°} = 500^2 + \frac{h^2}{\tan^2 39°}$$

$$h^2\left(\frac{1}{\tan^2 27°} - \frac{1}{\tan^2 39°}\right) = 500^2$$

$$h = \frac{500}{\sqrt{\frac{1}{\tan^2 27°} - \frac{1}{\tan^2 39°}}}$$

$= 327.8\ldots$

$= 328\text{m}$ (to nearest metre)

(d) $\frac{d}{dx}(\log_e 2x)$

$= \frac{2}{2x}$

$= \frac{1}{x}$

$\frac{d}{dx}(\log_e x)$

$= \frac{1}{x}$

$\therefore \frac{d}{dx}(\log_e 2x) = \frac{d}{dx}(\log_e x)$

No – as they differ by a constant

Q 6 (a) (i) $y = x^2$

$\frac{dy}{dx} = 2x$

at $x = t$

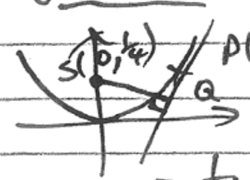
$\frac{dy}{dx} = 2t$

eq'n of tang at P is

$y - t^2 = 2t(x - t)$

$y = 2tx - t^2$ (2)

(ii)



$m_{SQ} = -\frac{1}{2t}$

eq'n of SQ is

$y - \frac{1}{4} = -\frac{1}{2t}(x - 0)$

$-2ty + \frac{1}{2}t = x$

$x = \frac{t}{2}(1 - 4y)$ — (1)

(iii) solving (1) & (2)

$y = 2t \times \frac{t}{2}(1 - 4y) - t^2$

$y = t^2 - 4t^2 y - t^2$

$y(1 + 4t^2) = 0$

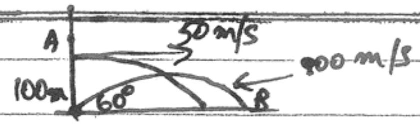
$y = 0$

$x = \frac{t}{2}$

$Q\left(\frac{t}{2}, 0\right)$

$\therefore$ locus of Q is the x axis.

(b)



$A: t = 0, x = 0, y = 100, \dot{x} = 50, \dot{y} = 0$

$B: t = 0, x = 0, y = 0, \dot{x} = 100\cos 60° = 50$

$\dot{y} = 100\sin 60° = 50\sqrt{3}$

A: $\ddot{x} = 0$ $\quad$ $\ddot{y} = -10$

$\dot{x} = c_1$ $\quad$ $\dot{y} = -10t + c_3$

$\dot{x} = 50$ $\quad$ $c_3 = 0$

$x = 50t + c_2$ $\quad$ $y = -5t^2 + c_4$

$c_2 = 0$ $\quad$ $c_4 = 100$

$x = 50t$ $\quad$ $y = -5t^2 + 100.$

B: $\ddot{x} = 0$ $\quad$ $\ddot{y} = -10$

$\dot{x} = c_1$ $\quad$ $\dot{y} = -10t + c_3$

$c_1 = 50$ $\quad$ $c_3 = 50\sqrt{3}$

$\dot{x} = 50$ $\quad$ $\dot{y} = -10t + 50\sqrt{3}$

$x = 50t + c_2$ $\quad$ $y = -5t^2 + 50\sqrt{3}t + c_4$

$c_2 = 0$ $\quad$ $c_4 = 0$

$x = 50t$ $\quad$ $y = -5t^2 + 50\sqrt{3}t$

to collide x & y must be equal at the same time.

$y = -5t^2 + 100 = -5t^2 + 50\sqrt{3}t$

$t = \frac{2}{\sqrt{3}}$ sec

if $t = \frac{2}{\sqrt{3}}$ $\quad$ $x = 50 \times \frac{2}{\sqrt{3}} = \frac{100}{\sqrt{3}}$

$y = -5 \times \left(\frac{2}{\sqrt{3}}\right)^2 + 100$

$= 93\frac{1}{3}$

$\therefore$ particles collide after $\frac{2}{\sqrt{3}}$ secs at $\left(\frac{100}{\sqrt{3}}, 93\frac{1}{3}\right)$

Q7 (a) $\sin x = 3\cos(x+65°)$

$\sin x = 3\cos x.\cos 65° - 3\sin x.\sin 65°$

$\sin x(1+3\sin 65°) = 3\cos x.\cos 65°$

$\tan x = \dfrac{3\cos 65°}{1+3\sin 65°}$ $(\cos x \neq 0, x \neq 90°)$

$\therefore x = 18°50', 198°50'$

(b) $A = \frac{1}{2}ab\sin C$

$\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

$b = \dfrac{a\sin B}{\sin A}$

$\therefore$ Area $= \frac{1}{2} a \times \dfrac{a\sin B}{\sin A}$

$= \dfrac{a^2\sin B}{2\sin A}$

(c) (i) $\lambda = -4$

(a) (ii) $y = \frac{x^2+5}{x+2} \quad (x \neq -2)$

$y \neq 0$ as $x^2+5 \neq 0$

$x = 0, \ y = \frac{5}{2}$

$$y' = \frac{(x+2)2x - (x^2+5)\cdot 1}{(x+2)^2}$$

$$= \frac{2x^2+4x-x^2-5}{(x+2)^2}$$

$$= \frac{x^2+4x-5}{(x+2)^2}$$

for max or min $y'=0$

$x^2+4x-5 = 0$

$(x+5)(x-1) = 0$

$x = -5$ or 1

$x=-5$ LHS $x=-6$ $y'>0$

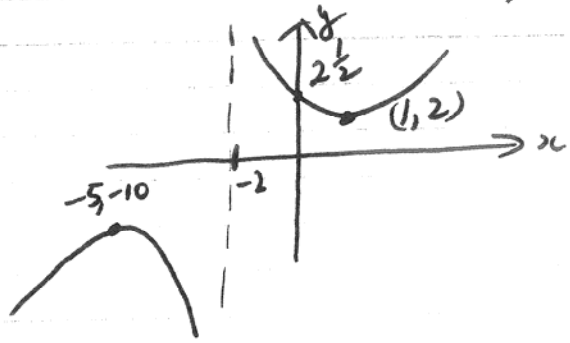
RHS $x=-4$ $y'<0$

$\therefore$ rel max at $(-5,-10)$

$x=1$ LHS $x=0$ $y'<0$

RHS $x=2$ $y'>0$

$\therefore$ rel min at $(1,2)$



$\therefore \ y \geq 2$ or $y \leq -10$.

(d) $mx = \frac{2x^2+1}{2x+4}$

$2mx^2+4mx = 2x^2+1$

$x^2(2m-2)+4mx-1=0$

equal roots

$16m^2-4(2m-2)x-1=0$

$16m^2+8m-8=0$

$2m^2+m-1=0$

$(2m-1)(m+1)=0$

$\therefore \ m = \frac{1}{2}$ or -1.